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Discussion of the condition 'monics are equalizers' implies 'epis are surjectives' in categories

by

Syed A. HUQ.

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0. Introduction

It seems a difficult question to determine the necessary and sufficient conditions for epimorphisms to be surjective in categories. Many people like Burgess [1], Reid [5], Eilenberg-Moore (cf. Mitchell [4], exercise 13 chapter I), Knauer [3] and Kurosh-Livshits-Schulgeifer [6] (cf. §6, section 5) have determined various cases of concrete categories in which epimorphisms are surjectives. The aim of this note is to point out a common property which makes the result work in all such categories. We also discuss why the result should not hold in general.

1. Main results

Definition 1.1. *A regular epimorphism* is a map which is a coequalizer of some pair of maps.

Definition 1.2. We shall say that a map α, *admits an image*, if α admits a factorization $\alpha=\nu\mu$, making the triangle

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\nu"'] & & B \\ & I \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutative, where ν is a regular epimorphism followed by the monomorphish μ. We say the image is *proper*, if

$$(I, \mu) \neq (B, 1_B)$$

THEOREM 1.3. *If in some category every monomorphism is an equalizer, then an epimorphism cannot admit proper image.*

Proof. Let $\delta: A \to B$ be an epimorphism. If it admits an image (ν, I, μ) then μ is an equalizer as well as an epimorphism. This implies $(I, \mu)=(B, 1_B)$. So (ν, I, μ) could not be a proper image.

Immediate is the

COROLLARY 1.4. *In a category in which every morphisms admits* *an image and every monomorphism is an equalizer, then every epimorphism is regular.*

So the question when are monomorphisms equalizers arises. We include the known result with proof for reader's convenience.

THEOREM 1.5. *In a finitely cocomplete category, a monomorphism $\mu: A \to B$ is an equalizer, if and only if μ is the equalizer of $(\tau_1\sigma, \tau_2\sigma)$ where (σ, D) is the coequalizer or $\mu\tau_1$, $\mu\tau_2$, where further τ_1, τ_2 are the canonical map in the coproduct diagram*

$$B \xrightarrow{\tau_1} B * B \xleftarrow{\tau_2} B$$

Proof. We only prove that if μ, happen to be an equalizer then it must be the equalizer of $\tau_1\sigma$, $\tau_2\sigma$, since the other part is obvious.

We consider the diagram

$$\begin{tikzcd}
 & & C & \\
A \arrow[r, "\mu"] & B \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] \arrow[r, shift left, "\tau_1"] \arrow[r, shift right, "\tau_2"'] & B*B \arrow[u, "\langle\alpha, \beta\rangle"'] \arrow[r, "\sigma"] & D \arrow[ul, "\varepsilon"']
\end{tikzcd}$$

where μ being regular is the equalizer of some pair $\alpha, \beta: B \to C$. Now the morphisms α, β determine a unique map $\langle\alpha, \beta\rangle: B * B \to C$, such that $\tau_1\langle\alpha, \beta\rangle = \alpha$ and $\tau_2\langle\alpha, \beta\rangle = \beta$. Since $\mu\alpha = \mu\beta$ implies $\mu\tau_1\langle\alpha, \beta\rangle = \mu\tau_2\langle\alpha, \beta\rangle$ and σ is the coequalizer of $\mu\tau_1$, $\mu\tau_2$, there exists an $\varepsilon: D \to C$ such that $\sigma\varepsilon = \langle\alpha, \beta\rangle$. Now if $\gamma\tau_1\sigma = \gamma\tau_2\sigma$, holds for an arbitrary γ, then $\gamma\alpha = \gamma\beta$, so $\gamma = \gamma'\mu$.

COROLLARY 1.6. *If the above finitely cocomplete category admits finite products, then a subobject (A, μ) is an equalizer if and only if the subobject $(A, \{\mu, \mu\tau_1\sigma, \sigma\})$ of $B \times D$ is the intersection of the subobjects $(B, \{1_B, \tau_1, \sigma\})$ and $(B, \{1_B, \tau_2\sigma\})$ of $B \times D$.*

Proof. For the definition of the intersection see Mitchell [4]. Now the rest follows from the fact that μ is equalizer of $\tau_1\sigma$, $\tau_2\sigma$ if and only if

$$\begin{tikzcd}
A \arrow[r, "\mu"] \arrow[d, "\mu"'] & B \arrow[d, "\{1_B. \tau_1\sigma\}"] \\
B \arrow[r, "\{1_B, \gamma_2\sigma\}"'] & B \times D
\end{tikzcd}$$

is a pullback diagram.

MAIN THEOREM 1.7. *In an equational category, where every monomorphism is an equalizer, every epimorphism is surjective.*

Proof. Since an equational category admits images, by corollary 1.4 every epimorphism is a coequalizer, as such surjective. We refer the reader to Barr [2_B] for the proof of the fact that coequalizers are surjective maps in equational categories.

Remark. For any triple T, the category $\mathscr{S}^T$ of T-algebras over $\mathscr{S}$, where $\mathscr{S}$ is the category of sets, has the property of epimorphisms being surjectives when monomorphisms are equalizers. The proof of this fact is exactly in the similer line as above.

2. Applications

In this section, we propose to show how the various proofs of the fact that epimorphisms are surjectives in different concrete categories appearing in the literature as mentioned in the introduction can be obtained as as a direct application of our theorem 1.7 instead of the complicated proof mentioned for different special cases.

2.1 *SETS.* For any subset $A \subset B$ take a set with two elements $\{0, 1\}$. Define $\alpha, \beta: B \to \{0, 1\}$ by $\alpha = \chi_A$, the characteristic function of A and β to be the constant function having value 1. The natural inclusion of A in B is an equalizer of α, β.

2.2 *GROUPS.* See the proof of Eilenberg-Moore given in Mitchell [4] which is really the proof of the fact that every monomorphism is an equalizer.

2.3 *A-acts.* See Knauer [3] for the proof of the fact that in the category of A-acts every monomorphism is an equalizer.

Hence in all these categories, every epimorphism must be surjective. Similer consideration holds in many familiar categories viz Toposes.

3. Discussion and counterexamples

3.1 If in the definition of the image we take ν to be an epimorphism and not necessarily a coequalizer, one can produce a counter example to show that in a concrete category, an epimorphism which admits an image and not a proper image need not be surjective, though monics may be equalizers.

Let $\mathscr{C}$ be a category whose objects are all finite sets with more than one element and one infinite set A. As morphisms we take all surjective maps between the finite sets, all maps form a finite set into A whose set theoretic image has precisely one element and the identity map of A. Then the monomorphisms of $\mathscr{C}$, are precisely the isomorphisms between the finite sets which are objects of $\mathscr{C}$, and the identity of A. Hence all of them are equalizers. All morphisms are epimorphisms. Each of them admits an image namely the identity of the codomain. However the maps in $\mathscr{C}$, from a finite set into A are obviously not surjective.

Remark. Theorem 1.7 works in an equational category only because the underlying faithful functor to the category of sets transports epimorphisms to surjective maps. All faithful functor certainly do not behave like this. Take for example.

Ab, to be the category of abelian groups, Z the integers and $Z_2 = Z/2Z$. Consider the the functor

$$(Z \oplus Z_2, \_) : Ab \rightarrow SET : G \mapsto (Z \oplus Z_2, G)$$

This is faithful, but most epis do not go to surjections for example the epimorphism $Z \rightarrow Z_2$ is transported by this functor to $(Z \oplus Z_2, Z) \rightarrow (Z \oplus Z_2, Z_2)$ i.e $Z \rightarrow Z_2 \oplus Z_2$ since Z is a generator.

3.2 We proceed to construct another example to show that our condition is not necessary i.e epis are surjectives, do not imply monics are equalizers.

We take $\mathscr{C}$ to be the following category with objects and arrows, as in the diagram

$$\begin{tikzcd}
Z \times Z \arrow[dr, "e"] \arrow[drr, "c"] & & \\
& R \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & N \\
Z \arrow[ur, "f"'] \arrow[urr, "d"'] & &
\end{tikzcd}$$

where N is the natural numbers, Z the integers and R the real numbers.

$$\begin{aligned}
&a(x) = \{\max |y| \mid y \in Z \quad \text{and} \quad y \leqq x\} \\
&b(x) \quad \{\min |y| \mid y \in Z \quad \text{and} \quad x \leqq y\} \\
&f(x) = x \\
&e(x, y) = x + y \\
&c(x, y) = ae(x, y) = a(x+y) = b(x+y) \\
&d(x) = a(x) = b(x)
\end{aligned}$$

Now we have

(i) a, b, c, d are epimorphisms and surjective maps.

(ii) e, f are not epimorphisms since $a \neq b$, but $ae = be$ and $af = bf$.

Thus any epimorphism is surjective; all the morphisms are monomorphisms and none of them are equalizers. Further any map α admits an image as $\alpha = \alpha \cdot 1$ where 1 is a cokernel and α is a monomorphism. Thus epis are surjectives do not imply monics are equalizers.

One can construct examples to show that epis are coequalizers even do not imply that monics are equalizers.

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Department of Mathematics,
Faculty of Science,
P.O. Box 1247.
Sana'a University.
Yemen Arab Republic